Optimal Fixed-Price Mechanism with Signaling

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ABSTRACT

Consider there is one seller and multiple buyers in a trade market. The seller aims to sell an indivisible item and maximize his revenue. When there is no information asymmetry between the seller and buyers, the seller can only design a fixed price mechanism. In this paper, we introduce information asymmetry between buyers and sellers, and study the mechanism design problem within a broader space, called the fixed-price signaling mechanism.

We assume the item has a quality, and buyers' valuation of the item depends on the quality of the item. The seller can privately observe the item's quality while buyers can only see its distribution. According to the theorem of Bayesian persuasion, the seller can influence buyers' valuations by strategically disclosing information about item quality, thereby adjusting his pricing strategies.

We consider two types of behavior models of buyers: ex-post individual rational (IR) buyers and ex-interim individual rational buyers. We show that when the market only has one buyer, the revenue generated from the optimal fixed price signaling mechanism is the same as that of the fixed price mechanism, no matter which behavior pattern the buyer uses. Furthermore, when there are multiple buyers in the market and all of them are ex-post IR, we show that there is no fixed price mechanism that is obedient for all buyers. However, if all buyers are ex-interim IR, we show that the optimal fixed-price signaling mechanism will bring more revenue for the seller than the fixed-price mechanism.

CCS CONCEPTS

• **Theory of computation → Algorithmic game theory and mechanism design**.

KEYWORDS

Information design, Mechanism design, Fixed-price mechanism

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1 INTRODUCTION

The phenomenon of information asymmetry is commonly seen in the real world and has attracted extensive research attention from both computer science and economics, including security [12, 14], advertising [1, 9], and voting [5]. In these applications, one agent with more information, called the sender, can influence the behavior of other agents by strategically disclosing information. Such an interactive process is usually modeled by the Bayesian persuasion framework [11], also known as information design [4]. In these information asymmetric scenarios, the sender is often in a leading position and can earn a higher utility.

There is also a line of work that introduces information design to auction scenarios [10, 13], or under specific auction formats, such as second-price auction [3] and posted price auction [7]. Different from previous settings, we introduce the information asymmetry into a trade market and restrict the design space to a simple format – the fixed-price mechanism. In this paper, we aim to study the influence of information asymmetry on the fixed-price mechanism design problem of the seller, and answer the following questions:

(1) What is the optimal mechanism when the seller can design information within the fixed price mechanism?
(2) Whether allowing the seller to design information can lead to higher revenues?

Specifically, we consider a trade market with one seller and multiple buyers. The seller aims to sell an indivisible item which has a quality. We assume the seller can privately observe the item's quality while buyers can only see the public distribution. Each buyer's valuation of the item depends on the item's quality. With the information advantage over the item's quality, the seller is able to design information before deciding on the fixed price.

1.1 Our Contributions

Throughout this paper, we consider two kinds of behavior patterns of buyers: the ex-post individual rational buyers and the ex-interim individual rational buyers. Moreover, we focus on two types of mechanism space: the fixed-price mechanism and the fixed-price signaling mechanism.

Optimal fixed-price mechanism. As a benchmark, we first investigate the optimal fixed-price mechanism design problem for the seller. For ex-post IR buyers, the seller needs to balance the trade price and the trade probability of the item when setting the fixed price. But for ex-interim IR buyers, the seller can raise the trade price as much as possible while ensuring that at least one buyer is willing to buy the item.

Optimal fixed-price signaling mechanism for single buyer. As a warm-up, we also consider a special case of our original setting. We study the optimal fixed-price signaling mechanism when there is only one buyer in the market. We are surprised to find that the revenue generated from the optimal fixed-price signaling mechanism is the same as that of the fixed-price mechanism, no matter which behavior patterns the buyer uses. It means that when there is only one buyer in the market, allowing the seller to design information will not bring him more revenue and the advantage in information does not translate into an advantage in revenue.

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Optimal fixed-price signaling mechanism for multiple buyers. Furthermore, we study the optimal fixed-price signaling mechanism design problem when there are multiple buyers in the market. For ex-post IR buyers, we find that there is no fixed-price signaling mechanism that is obedient for all the buyers. But if we assume that buyers cannot buy the item when the seller does not recommend it, we show that this problem can be solved in closed form. As for ex-interim IR buyers, we show that the optimal fixed-price signaling mechanism will bring more revenue for the seller than the fixed-price mechanism.

1.2 Related Work

Information design. Our research is grounded in the literature on information design. We adopt the "Bayesian persuasion" framework, proposed by the seminal work [11], to model how the seller designs information. Like most follow-up work, we assume there is only one side that has private information. One of the related works in this line is the model proposed by Castiglioni et al. [6], where an informed sender persuades a set of uninformed receivers. However, there is a substantial distinction between persuasion and selling an item through persuasion. In our context, the seller has only one item for sale, so even if many buyers are willing to buy the item, only one buyer will eventually get the item.

Joint design mechanism and information. Our work aligns with the research on the combination of mechanism design with information design. Eső and Szentes [10] also consider a setting where an item seller sells one indivisible item to multiple buyers. However, in their model, the seller cannot observe the item's quality. Another difference lies in the mechanism space, they do not restrict design space while we focus on the fixed-price mechanism. Wei and Green [13] study a single buyer setting under a general design space and give a closed-form solution. In contrast, we consider there are multiple buyers in the market. Closer to us are the series of works that focus on information design under specific auction formats, such as second-price auction [3] and posted price auction [7]. Castiglioni et al. [7] consider a posted price auction where buyers arrive sequentially and their valuations for the item depend on a random state that is only observed by the seller. However, the key difference between their model and ours is that in their model, the seller's price function depends on the signal sent by the seller but we focus on a constant price.

The rest of the paper is structured as follows. Section 2 describes the model and two types of mechanism space considered in this paper. Section 3 investigates the optimal fixed-price mechanism without signaling. In section 4, we first study the optimal fixed-price signaling mechanism design problem under one buyer setting and later generalize it to the multi-buyer setting. We summarize all the results in Section 5. Finally, we conclude in Section 6.

2 PRELIMINARIES

2.1 Model

Consider a trade market with one seller and multiple buyers. The seller has an indivisible item for sale and buyers want to buy it. We assume that the item has a quality q, which is drawn from a continuous distribution with both cumulative distribution function (CDF) $G(q)$ and probability density function (PDF) $g(q)$ of a support set $[q_1, q_2]$. The seller can observe the item's quality privately, but the corresponding distribution $G(q)$ is public for all the buyers.

We assume that all buyers have no private information, so their valuation of the item only depends on its quality. Formally, let N denote the set of buyers and $v_i : Q \mapsto \mathbb{R}^+$ be the valuation function of buyer i. That is, the buyer perceives the item's value to be $v_i(q)$ when the item's quality is q. The higher the quality of the item, the higher the buyer's valuation of the item should be. So we also assume that $v_i(q)$ is monotone increasing with respect to q, for all i. Then we can define the inverse function of $v_i(\cdot)$ as $v_i^{-1}(\cdot)$.

When there is no additional information about the item's q, each buyer only has a prior valuation on the item based on their prior belief over q. Specifically, the prior valuation of buyer i can be denoted by $\mathbf{E}_{q\sim G(q)}[v_i(q)]$.

Stand on the side of the seller, we aim to design a revenue-maximizing item-selling mechanism for the seller. In addition, we assume that the seller cannot adopt price-discriminating mechanisms and also the price does not depend on the quality of the item. Therefore, the seller can only set a fixed price for the item, denoted by p.

Individual Rationality (IR). We consider two types of buyers: the ex-post IR buyer and the ex-interim IR buyer. Below, we give the definition of these two types of buyers.

Definition 2.1 (Ex-post IR). An ex-post IR buyer will buy the item if and only if his valuation on the item is no less than p after knowing the actual q. Formally, it is equivalent to:

$$v_i(q) \geq p.$$

We can also define the ex-post utility of the buyer as follows:

$$U_{post}(\pi, p) = v_i(q) - p$$

Definition 2.2 (Ex-interim IR). An ex-interim IR buyer will buy an item if and only if his expected valuation on the item is no less than p. Formally, it is equivalent to:

$$\mathop{\mathbf{E}}_{q}[v_i(q)] \geq p.$$

Similarly, we define the ex-interim utility of the buyer as follows:

$$U_{interim}(\pi, p) = \mathop{\mathbf{E}}_{q}[v_i(q)] - p$$

2.2 Mechanism Space

Before designing the mechanism for the seller, we first need to describe in what space the seller can select the mechanism. Since we do not allow price discrimination, we mainly focus on two kinds of fixed-price mechanisms: the fixed-price mechanism and the fixed-price signaling mechanism.

Fixed-price Mechanism. We first introduce a simple class of mechanisms, called the fixed-price mechanism. In a fixed-price mechanism, the seller assigns a price of p to the item, and when the item is sold, the buyer pays p to the seller.

In a fixed-price mechanism, the interaction between the seller and buyers takes place as follows:

(1) The seller observes the item's quality q, and the buyers observe the distribution $G(q)$.

(2) The seller sets a price p for the item and announces it to all buyers.
(3) If at least one buyer is willing to buy the item[1], then the item is sold and the seller receives p.
(4) Otherwise, the seller keeps the item.

Fixed-price Mechanism with Signaling . As indicated in step (1) of the fixed price mechanism, in our setting, the seller has more information about the item's quality than buyers. According to the well-known Bayesian Persuasion theory [11], the seller can design information before deciding on the price, enlarging the seller's design space. Within this new space, we refer to it as the fixed-price mechanism with signaling, the optimal mechanism must ensure that the seller's revenue is at least as high as that achieved by the fixed-price mechanism.

Information design. When there is information asymmetry in the market and that information affects buyer behavior, the seller can influence buyers' beliefs about the quality of the item by strategically disclosing information, which is also known as information design. Following the "Bayesian Persuasion" framework, the seller can disclose information by way of signaling. Specifically, the seller can first commit to a signaling scheme, which is a mapping from the quality set to a distribution over a signal set. Then, after observing the item's quality, he will send a signal to buyers based on the committed signaling scheme.

Belief update. After receiving signal, each buyer will update their belief over q by the bayes update rule.

Now, we are ready to formally describe the set of fixed-price signaling mechanisms.

Definition 2.3 (Fixed-price signaling mechanism). A fixed-price signaling mechanism $\mathcal{M}$ can be described by a tuple (π, p), where:

- $\pi : Q \mapsto \Delta(\Sigma)$ is the signaling scheme. When the item's quality is q, the seller will send signal $\sigma \in \Sigma$ with probability $\pi(q, \sigma)$.
- p is the fixed price. When the item is sold, the buyer pays p to the seller.

In a fixed-price signaling mechanism, the interaction between the seller and buyers takes place as follows:

(1) The seller announces the mechanism $\mathcal{M} = (\pi, p)$ to all buyers.
(2) The seller observes the item's quality q, and the buyers observe the distribution $G(q)$.
(3) The seller sends signal $\sigma \in \Sigma$ drawn from distribution $\pi(q, \cdot)$
(4) Buyers update their belief and then decide whether to buy.
(5) If at least one buyer is willing to buy, then the item is sold and the seller receives p.

In this paper, we aim to design a fixed-price signaling mechanism that maximizes the seller's expected revenue.

[1] If multiple buyers are willing to buy the item, the seller selects one among them randomly.

3 FIXED-PRICE MECHANISM WITHOUT SIGNALING

In this section, we design the optimal fixed price for the seller under the fixed-price mechanism.

3.1 When the buyer is ex-post IR

For ex-post IR buyers, the higher the price of an item, the lower the probability that a buyer will buy it. Therefore, when designing the fixed price mechanism, the seller needs to balance the probability of the item being sold and the price of the item.

Proposition 3.1. *When buyers are ex-post IR, to maximize the expected revenue, the seller should set the fixed price as follows:*

$$p^* \in \arg\max_p Rev_{fix}(p),$$

where $Rev_{fix}(p) = [1 - \prod_{i \in N} G(v_i^{-1}(p))] \cdot p$ *denotes the expected revenue from setting a fixed price* p.

Proof. For ex-post IR buyers, they will only be willing to buy the item if the valuation is greater than or equal to p. Without signaling, they will not get additional information about the quality of the item, so they can only make decisions based on the prior belief $G(q)$. The probability that buyer i's valuation is less than p equals:

$$Pr\{v_i(q) < p\} = Pr\{q < v_i^{-1}(p)\} = G(v_i^{-1}(p)).$$

Then the probability that all buyers' valuation is less than p is $\prod_{i \in N} G(v_i^{-1}(p))$, which also denotes the probability that the item cannot be sold. So the probability that at least one buyer is willing to buy is $1 - \prod_{i \in N} G(v_i^{-1}(p))$.

Based on the above analysis, the seller's expected revenue from setting a fixed price p can be represented as:

$$Rev_{fix}(p) = [1 - \prod_{i \in N} G(v_i^{-1}(p))] \cdot p.$$

Then the seller can optimally set the price:

$$p^* \in \arg\max_p Rev_{fix}(p).$$

This concludes the proof. □

3.2 When the buyer is ex-interim IR

Without additional information about q, buyer i's expected valuation $v_i(q)$ depends on the prior distribution $G(q)$, which is a constant. So the seller can raise the price as much as they can while ensuring that at least one buyer will buy the item.

Proposition 3.2. *When buyers are ex-interim IR, to maximize the expected revenue, the seller should set the fixed price as:*

$$p^* = \max_i \bar{v}_i,$$

where $\bar{v}_i = \mathbf{E}_{q \sim G(q)}[v_i(q)]$ *is the prior valuation of buyer* i *on the item.*

Proof. For ex-interim IR buyers, they will only be willing to buy the item if their expected valuation of the item is greater than or equal to p.

Note that without signaling, the valuation of buyer i on the item is a constant:

$$\bar{v}_i = \mathop{\mathbf{E}}_{q \sim G(q)}[v_i(q)].$$

Thus the buyer i will not be willing to buy the item if the fixed price $p > \bar{v}_i$. Moreover, there is no buyer will buy the item if $p > \max_i \bar{v}_i$.

We denote by $\bar{p} = \max_i \bar{v}_i$. Then the seller's expected revenue from setting a fixed price p can be written as:

$$Rev_{fix}(p) = \mathbb{I}\{p <= \bar{p}\} \cdot p,$$

where $\mathbb{I}\{\cdot\}$ is an indicator function. It is easy to find that to maximize the revenue, the fixed price should be set at $\bar{p}$, that is:

$$\bar{p} = \arg\max_p Rev_{fix}(p)$$

This concludes the proof. □

4 FIXED-PRICE MECHANISM WITH SIGNALING

In section 3, we discuss the optimal mechanism within the fixed-price mechanism, where the seller can only design a fixed price. In this section, we consider a more general mechanism space – the fixed-price signaling mechanism, which allows the seller to design information before deciding on the fixed price.

4.1 Warm-up: Only One Buyer in the Market

In this section, as a warm-up, we consider a special case of our general model where there is only one buyer in the market. Therefore, the seller only needs to design one player's information. In this case, each signal $\sigma \in \Sigma$ only needs to be one dimensional.

After receiving signal σ, the buyer will update his belief about q and decide whether to buy the item. According to the revelation principle, it is without loss of generality to view each signal as an action recommendation, as each signal will induce a posterior belief, which leads to a certain action [8, 11]. In our setting, there are only two actions for the buyer: buy or not buy, thus we only need two signals in set Σ. We say a mechanism is obedient if the buyer will always follow the action recommendation.

Definition 4.1 (Obedience). A mechanism (π, p) is obedient if the buyer has no incentive to deviate from the action recommendation from the seller.

Let $\Sigma = \{0, 1\}$, where signal 1 for buy and signal 0 for not buy. For simplicity, we use $\pi(q)$ to denote the probability of sending signal 1 when the item's quality is q. Naturally, $1 - \pi(q)$ denotes the probability of sending signal 0.

Once receiving signal 1, the buyer will update his belief over q as follows:

$$g(q|1) = \frac{\pi(q) \cdot g(q)}{\int_{q' \in Q} \pi(q') \cdot g(q')\, \mathrm{d}q'}. \tag{1}$$

Similarly, after receiving signal 0, the posterior belief of the buyer over q is:

$$g(q|0) = \frac{[1 - \pi(q)]g(q)\, \mathrm{d}q}{\int_{q' \in Q} [1 - \pi(q')]g(q')\, \mathrm{d}q'}. \tag{2}$$

Given an obedient mechanism (π, p), the seller's revenue can be written as:

$$Rev_{sig}(\pi, p) = \int_q \pi(q) g(q)\, \mathrm{d}q \cdot p. \tag{3}$$

Next, we discuss what constraints should be satisfied by an obedient mechanism and what is the optimal mechanism within the obedient mechanism, for ex-post IR buyers and ex-interim IR buyers, respectively.

4.1.1 Ex-post IR buyer. We first discuss what constraints should an obedience mechanism satisfy when facing an ex-post buyer.

Specifically, to ensure the obedience of the buyer, we need to pose two constraints on the mechanism (π, p): (1) after receiving signal 1, the buyer's ex-post utility from buying the item is at least 0; (2) after receiving signal 0, the buyer's ex-post utility from buying the item is at most 0.

When receiving signal 1, the buyer obtains posterior belief $g(q|1)$ based on Equation (1). Given the posterior belief $g(q|1)$, we derive the probability that buyer i's valuation is less than p as follows:

$$\begin{aligned} Pr\{v(q) < p\} =& Pr\{q < v^{-1}(p)\} \\ =& \int_{q_1}^{v^{-1}(p)} g(q|1)\, \mathrm{d}q \\ =& \frac{\int_{q_1}^{v^{-1}(p)} \pi(q) \cdot g(q)\, \mathrm{d}q}{\int_{q'} \pi(q') \cdot g(q')\, \mathrm{d}q'}. \end{aligned}$$

Then the probability that the buyer is willing to buy the item is $1 - Pr\{v(q) < p\}$. Now we only need to ensure that after receiving signal 1, the probability $1 - Pr\{v(q) < p\} = 1$, that is:

$$\int_{q_1}^{v^{-1}(p)} \pi(q) \cdot g(q)\, \mathrm{d}q = 0. \tag{4}$$

Similarly, after receiving signal 0, the buyer derives posterior belief $g(q|0)$ based on Equation (2). Given this, the probability that the buyer's ex-post valuation is less than p is:

$$\begin{aligned} Pr\{v(q) < p\} &= \int_{q_1}^{v^{-1}(p)} g(q|0)\, \mathrm{d}q \\ &= \frac{\int_{q_1}^{v^{-1}(p)} [1 - \pi(q)]g(q)\, \mathrm{d}q}{\int_{q'} [1 - \pi(q')]g(q')\, \mathrm{d}q'} \end{aligned} \tag{5}$$

Then we need to ensure that after receiving signal 0, the Equation (5) is equal to 1, that is:

$$\int_{q_1}^{v^{-1}(p)} [1 - \pi(q)]g(q)\, \mathrm{d}q = \int_Q [1 - \pi(q)]g(q)\, \mathrm{d}q.$$

We can also call these obedience constraints as *ex-post obedience* constraints.

Theorem 4.2. *When the buyer is ex-post IR, the following signaling scheme π^* and fixed price p^* forms an optimal fixed price signaling mechanism:*

$$\pi^*(q) = \begin{cases} 0 & if\, q < v^{-1}(p) \\ 1 & otherwise \end{cases},$$

$$p^* = \arg\max_p [1 - G(v^{-1}(p))] \cdot p.$$

Proof. According to obedience constraint (4), we should set $\pi(q)$ to zero when q is between q_1 and $v^{-1}(p)$, that is:

$$\pi(q) = 0 \text{ if } q_1 \le q < v^{-1}(p).$$

And we can rewrite obedience contain (5) as follows:

$$\int_{v^{-1}(p)}^{q_2} [1 - \pi(q)]g(q)\, dq = 0.$$

Thus we should set $\pi(q) = 1$ when q is between $v^{-1}(p)$ and q_2. Overall, we obtain the optimal signaling scheme as follows:

$$\pi^*(q) = \begin{cases} 0 & \text{if } q_1 \le q < v^{-1}(p) \\ 1 & \text{if } v^{-1}(p) \le q \le q_2 \end{cases}.$$

Then given the signaling scheme π^*, the expected revenue of the seller can be written as:

$$\begin{aligned} Rev_{sig}(\pi^*, p) &= Pr\{\pi^*(q) = 1\} \cdot p \\ &= Pr\{q \ge v^{-1}(p)\} \cdot p \\ &= [1 - G(v^{-1}(p))] \cdot p. \end{aligned}$$

So the optimal fixed price can be derived by:

$$p^* \in \arg\max_p Rev_{sig}(\pi^*, p)$$

This concludes the proof. □

It is surprising that the optimal revenue obtained within the fixed price signaling mechanism is the same as that obtained within the fixed price mechanism space.

Proposition 4.3. *When facing an ex-post IR buyer, the optimal revenue obtained in the fixed price signaling mechanism is equal to that obtained in the fixed price mechanism.*

It means that in this setting, allowing the seller to design information before setting a price does not lead to higher returns for him.

4.1.2 Ex-interim IR buyer.

Recall that without additional information, the buyer's expected valuation on the item is a constant. But when the seller can design information, he can influence the buyer's expected valuation via signaling. Next, we discuss what constraints should an obedient mechanism satisfy when facing an ex-interim IR buyer.

Given posterior belief $g(q|1)$, the ex-interim valuation of the buyer on the item is equivalent to:

$$\mathop{\mathbf{E}}_{q \sim g(q|1)}[v(q)] = \int_Q v(q)g(q|1)\, dq.$$

So to ensure the buyer is willing to buy the item after receiving signal 1, we need to require that the ex-interim utility of the buyer is no less than 0, that is:

$$\int_q g(q|1)v(q)\, dq - p \ge 0. \tag{6}$$

With some simple algebraic manipulations, we obtain:

$$\int_q \pi(q)[v(q) - p]g(q)\, dq \ge 0. \tag{7}$$

Similarly, after receiving signal 0, the valuation of the buyer is:

$$\mathop{\mathbf{E}}_{q \sim g(q|0)}[v(q)] = \int_Q v(q)g(q|0)\, dq.$$

To ensure the buyer will not buy the item after receiving signal 0, we need to require that:

$$\int_q v(q)g(q|0)\, dq - p \le 0.$$

With some simple algebraic manipulations, we obtain:

$$\int_q \pi(q)[v(q) - p]g(q)\, dq \ge \mathop{\mathbf{E}}_{q \sim g(q)}[v(q)] - p. \tag{8}$$

We can also call these obedience constraints as ex-interim obedience constraints.

Combine with the seller's objective, we can formulate the optimal mechanism design problem as the following optimization program:

$$\begin{aligned} \max_{\pi, p} \quad & \int_Q \pi(q)g(q)\, dq \cdot p \\ \text{s.t.} \quad & \int_Q \pi(q)[v(q) - p]g(q)\, dq \ge 0 \\ & \int_Q \pi(q)[v(q) - p]g(q)\, dq \ge \mathop{\mathbf{E}}_q[v(q)] - p \end{aligned} \tag{9}$$

From the obedience constraint, we can obtain an upper bound of the program (9).

Proposition 4.4. *The optimization program* (9) *is upper bounded by:*

$$Rev_{sig}(\pi, p) \le \mathop{\mathbf{E}}_q[v(q)].$$

Proof. According to constraint (7), we have:

$$\begin{aligned} \int_q \pi(q)g(q)\, dq \cdot p &\le \int_q \pi(q)v(q)g(q)\, dq \\ &\le \int_q v(q)g(q)\, dq \\ &= \mathop{\mathbf{E}}_q[v(q)]. \end{aligned}$$

This concludes the proof. □

Next, we show that we can construct an obedience mechanism that can achieve this upper bound, thus achieving optimal.

Theorem 4.5. *When the buyer is ex-post IR, the following signaling scheme π and fixed price p forms an optimal fixed price signaling mechanism:*

$$\pi^*(q) = 1, \forall q \in Q \quad \text{and} \quad p^* = \mathop{\mathbf{E}}_q[v(q)].$$

Proof. The proof process is decomposed into two parts. First, we ignore constraint (8) and focus on the remaining optimization program. Second, we show that the optimal solution in the relaxed problem also satisfies constraint (8), thus remaining optimal in the original program (9).

Firstly, we raise the price p such that the obedience constraint (7) is tight, that is:

$$\int_Q \pi(q)v(q)g(q)\, dq = \int_Q \pi(q)g(q)\, dq \cdot p.$$

Put the price p on one side, we get:

$$p = \frac{\int_Q \pi(q)v(q)g(q)\,\mathrm{d}q}{\int_Q \pi(q)g(q)\,\mathrm{d}q}. \tag{10}$$

Constraint (7) ensures that the buyer will buy the item after receiving signal 1, so under this price, the buyer will buy the item after receiving signal 1. To maximize revenue, the seller should send signal 1 as much as possible, then we construct the following signaling scheme:

$$\pi^*(q) = 1, \forall q \in Q,$$

which means no matter what q the seller observes, he sends signal 1. Also, the buyer will always buy the item. Then Equation (10) becomes:

$$p^* = \int_Q v(q)g(q)\,\mathrm{d}q = \mathop{\mathbb{E}}_q[v(q)]$$

Next, we show the constructed mechanism (π^*, p^*) also satisfies the omitted constraint (8). Putting (π^*, p^*) into constraint (8), we obtain:

$$\begin{aligned}\int_Q [v(q) - p]g(q)\,\mathrm{d}q &= \int_Q v(q)g(q)\,\mathrm{d}q - p \\ &= \mathop{\mathbb{E}}_q[v(q)] - p.\end{aligned}$$

This concludes the proof. □

In hindsight, we can see that since $\pi = 1$ for all $q \in Q$, the probability of sending signal 0 is 0. That is why when solving the program (9), we can safely ignore constraint (8).

From the Theorem 4.5, we have the following observation.

Observation 1. *When facing an ex-interim IR buyer, the optimal revenue obtained in the fixed price signaling mechanism is equal to that obtained in the fixed price mechanism.*

Combining with Observation (4.3), we can conclude that when there is one buyer in the market, allowing the seller to design information before setting the fixed price will not bring him more revenue.

Proposition 4.6. *When there is one buyer in the market, allowing the seller to design information before setting the fixed price will not bring him more revenue.*

4.2 Multiple buyers in the Market

In this section, we consider the original model as described in Section 2.1, where there are n buyers in the market and the seller only has one indivisible item for sale.

In this case, the signal set Σ can be n-dimension, and the seller can send different signals to different buyers, leading to different posterior beliefs. The signal space will be very large, but with the help of the revelation principle, we can reduce the signal space.

Lemma 1 (Bergemann et al. [2]). *It is without loss of generality to focus on the responsive experiment where the signal space has at most the cardinality of the outcome space.*

Based on the above results, we can focus on the set of signaling schemes where a one-to-one correspondence exists between signals and outcomes. In our setting, there are $n + 1$ possible outcomes, with n of them corresponding to each buyer obtaining the item and an additional one corresponding to no buyer buying the item. Thus, we can define Σ as follows:

$$\Sigma = \left\{ \boldsymbol{\sigma} \in \{0,1\}^n : \sum_{i=1}^{n} \sigma_i \leq 1 \right\},$$

where $\boldsymbol{\sigma}$ with $\sigma_i = 1$ corresponds to the outcome where buyer i obtains the item and with $\sigma_i = 0$ for all i corresponds to the outcome where no buyer buys the item. From the implementation perspective, the seller can send the i-th element of $\boldsymbol{\sigma}$ to the buyer i, indicating whether the buyer should buy or not. For simplicity, we denote the signal with $\sigma_i = 1$ as s_i, and the signal with $\sigma_i = 0$ for all i as s_0. Therefore, sending signal s_i means the seller asks the buyer i to buy the item.

The seller has only a single item for sale, so π must satisfy the following constraints:

$$\sum_{i \in N} \pi(q, s_i) + \pi(q, s_0) = 1 \quad \text{and} \quad \pi(q, s_i) \geq 0, \forall i, \forall q. \tag{11}$$

Once receiving signal 1, buyer i will update belief over q by:

$$g(q|1) = \frac{\pi(q, s_i)g(q)}{\int_{q'} \pi(q', s_i)g(q')\,\mathrm{d}q'} \tag{12}$$

Similarly, once receiving signal 0, the posterior belief of buyer i becomes:

$$g(q|0) = \frac{[1 - \pi(q, s_i)]g(q)}{\int_{q'} [1 - \pi(q', s_i)]g(q')\,\mathrm{d}q'} \tag{13}$$

Whether it is multiple buyers or a single buyer, the seller's revenue is the probability of the item being sold times the fixed price, as indicated in Equation (3).

4.2.1 Ex-post buyers. We first discuss what constraints should an obedience mechanism satisfy when facing multiple ex-post buyers.

When receiving signal 1, the probability that buyer i's valuation is less than p is:

$$Pr\{v_i(q) < p \mid 1\} = \frac{\int_{q_1}^{v_i^{-1}(p)} \pi(q, s_i)g(q)\,\mathrm{d}q}{\int_{q'} \pi(q', s_i)g(q')\,\mathrm{d}q'}.$$

To ensure the buyer will buy the item after receiving signal 1, it is equivalent to $Pr\{v_i(q) < p \mid 1\} = 0$, that is:

$$\int_{q_1}^{v_i^{-1}(p)} \pi(q, s_i)g(q)\,\mathrm{d}q = 0.$$

Similarly, after receiving signal 0, the probability that the buyer's valuation is less than p is:

$$\begin{aligned}Pr\{v_i(q) < p \mid 0\} &= \int_{q_1}^{v_i^{-1}(p)} g(q|0)\,\mathrm{d}q \\ &= \frac{\int_{q_1}^{v_i^{-1}(p)} [1 - \pi(q, s_i)]g(q)\,\mathrm{d}q}{\int_{q'} [1 - \pi(q', s_i)]g(q')\,\mathrm{d}q'}.\end{aligned}$$

Then to ensure obedience, it is equivalent to $Pr\{v_i(q) < p \mid 0\} = 1$:

$$\int_{q_1}^{v_i^{-1}(p)} [1 - \pi(q, s_i)]g(q)\,\mathrm{d}q = \int_q [1 - \pi(q, s_i)]g(q)\,\mathrm{d}q,$$

which can also be rewritten as:

$$\int_{v_i^{-1}(p)}^{q_2} [1-\pi(q,s_i)]g(q)\,\mathrm{d}q = 0.$$

Proposition 4.7. *When facing multiple ex-post IR buyers, a mechanism (π, p) is ex-post obedience if and only if:*

$$\int_{q_1}^{v_i^{-1}(p)} \pi(q,s_i)g(q)\,\mathrm{d}q = 0, \forall i \in N \tag{14}$$

$$\int_{v_i^{-1}(p)}^{q_2} [1-\pi(q,s_i)]g(q)\,\mathrm{d}q = 0, \forall i \in N. \tag{15}$$

Theorem 4.8. *When multiple ex-post IR buyers are in the market, there is no fixed-price signaling mechanism that is obedience for all the buyers.*

Proof. According to the obedience constraint (14), we have:

$$\pi(q,s_i) = 0 \text{ if } q \in [q_1, v_i^{-1}(p)), \forall i \in N$$

And to satisfy constraint (15), we have:

$$\pi(q,s_i) = 1 \text{ if } q \in [v_i^{-1}(p), q_2], \forall i \in N$$

Overall, we obtain the following signaling scheme:

$$\pi(q,s_i) = \begin{cases} 1 & \text{if } i=0 \text{ and } q \in [q_1, v_{min}^{-1}(p)) \\ 0 & \text{if } i \in N \text{ and } q \in [q_1, v_{min}^{-1}(p)) \\ 0 & \text{if } i=0 \text{ and } q \in [v_{min}^{-1}(p), q_2] \\ 1 & \text{if } i \in N \text{ and } q \in [v_i^{-1}(p), q_2] \end{cases},$$

where $v_{min}^{-1}(p) = \min_{i\in N} v_i^{-1}(p)$.

Note that in the above signaling scheme, the seller sends s_i with probability 1 for all $i \in N$ when $q \in [v_i^{-1}(p), q_2]$, which contradicts the probability constraint (11). Therefore, there is no obedient fixed-price signaling mechanism, when facing multiple ex-post IR buyers. □

But if we assume that the buyer cannot buy the item when the seller does not recommend it, then we can omit constraint (15) and obtain the following results.

Theorem 4.9. *If buyers only are allowed to buy the item after receiving signal 1, the following mechanism (π^*, p^*) is one of the optimal one:*

$$\pi^*(q,s_i) = \begin{cases} 1 & \textit{if } i=0 \textit{ and } q \in [q_1, v_{min}^{-1}(p)) \\ 0 & \textit{if } i \in N \textit{ and } q \in [q_1, v_{min}^{-1}(p)) \\ 0 & \textit{if } i=0 \textit{ and } q \in [v_{min}^{-1}(p), q_2] \\ 1 & \textit{if } i=j \textit{ and } q \in [v_{min}^{-1}(p), q_2] \end{cases},$$

where $j = \arg\min_i v_i^{-1}(p)$.

$$p^* \in \arg\max_p [1-G(v_{min}^{-1}(p))] \cdot p.$$

Proof. Now the obedience constraint is fully characterized by constraint (14). To satisfy this constraint, we obtain the following signaling scheme:

$$\pi(q,s_i) = \begin{cases} 1 & \text{if } i=0 \text{ and } q \in [q_1, v_{min}^{-1}(p)) \\ 0 & \text{if } i \in N \text{ and } q \in [q_1, v_{min}^{-1}(p)) \\ 0 & \text{if } i=0 \text{ and } q \in [v_{min}^{-1}(p), q_2] \\ 1 & \text{if } i=j \text{ and } q \in [v_{min}^{-1}(p), v_{max}^{-1}(p)] \end{cases},$$

where $v_{max}^{-1}(p) = \max_{i\in N} v_i^{-1}(p)$.

As for $q \in [v_{max}^{-1}(p), q_2]$, all buyers will be willing to buy the item after receiving signal 1. So we can randomly choose a buyer, that is:

$$\sum_{i\in N} \pi(q,s_i) = 1.$$

Under this signal scheme, the seller's revenue becomes:

$$\begin{aligned} Rev_{sig}(\pi,p) &= Pr\{\sum_{i\in N}\pi(q,s_i)\} \cdot p \\ &= Pr\{q \in [v_{min}^{-1}(p), q_2]\} \cdot p \\ &= [1-G(v_{min}^{-1}(p))] \cdot p \end{aligned}$$

So the optimal fixed price can be derived by:

$$p^* \in \arg\max_p [1-G(v_{min}^{-1}(p))] \cdot p.$$

This concludes the proof. □

4.2.2 Ex-interim IR buyers. Next, we discuss what constraints should an obedient mechanism satisfy when facing multiple ex-interim IR buyers.

We need to ensure that the ex-interim utility of buyer i from buying the item is no less than 0 when receiving signal 1. Combine Equation (6) and (12), we get:

$$\int_q \pi(q,s_i)[v_i(q)-p]g(q)\,\mathrm{d}q \ge 0, \forall i \in N. \tag{16}$$

Similarly, to ensure buyer i will not buy the item after receiving signal 0, we have:

$$\int_q \pi(q,s_i)[v_i(q)-p]g(q)\,\mathrm{d}q \ge \mathop{\mathbf{E}}_{q\sim g(q)}[v_i(q)] - p, \forall i \in N. \tag{17}$$

Combine with the seller's objective, we can formulate the optimal mechanism design problem as the following program:

$$\begin{aligned} \max_{\pi,p} \quad & \int_Q \sum_{i\in N} \pi(q,s_i)g(q)\,\mathrm{d}q \cdot p \\ \text{s.t.} \quad & \int_q \pi(q,s_i)[v_i(q)-p]g(q)\,\mathrm{d}q \ge 0, \qquad \forall i \in N \\ & \int_q \pi(q,s_i)[v_i(q)-p]g(q)\,\mathrm{d}q \ge \mathop{\mathbf{E}}_q[v_i(q)] - p, \forall i \in N \end{aligned} \tag{18}$$

From obedience constraint (16), we can obtain the program's upper bound.

Proposition 4.10. *The program (18) is upper bounded by:*

$$Rev_{sig}(\pi,p) \le \mathop{\mathbf{E}}_q[v_{max}(q)],$$

where $v_{max}(q) = max_i v_i(q)$.

Proof. According to constraint (16), we have:

$$\int_q \pi(q,s_i)v_i(q)g(q)\,\mathrm{d}q \ge \int_q \pi(q,s_i)g(q)\,\mathrm{d}q \cdot p, \forall i \in N.$$

Table 1: A summary of the optimal revenue obtained from the fixed-price mechanism and the fixed-price signaling mechanism under different buyers' behavior patterns

Behavior pattern	Fixed-price mechanism	Fixed-price signaling with one buyer	Fixed-price signaling
Ex-post IR	$\max_p [1 - \prod_{i \in N} G(v_i^{-1}(p))] \cdot p$	$\max_p [1 - G(v^{-1}(p))] \cdot p$	$-$, $\max_p [1 - G(v_{min}^{-1}(p))] \cdot p$
Ex-interim IR	$\max_i \mathbf{E}[v_i(q)]$	$\mathbf{E}[v(q)]$	$\mathbf{E}[\max_i v_i(q)]$

Sum over all constraints, we obtain:

$$\int_q \sum_{i \in N} \pi(q, s_i) g(q) \,\mathrm{d}q * p \le \int_q \sum_{i \in N} \pi(q, s_i) v_i(q) g(q) \,\mathrm{d}q$$
$$\le \int_q v_{max}(q) g(q) \,\mathrm{d}q = \mathop{\mathbf{E}}_q [v_{max}(q)],$$

This concludes the proof. □

Next, we show that we can construct an obedience mechanism that achieves this upper bound, thus achieving optimal.

Theorem 4.11. *When there are multiple ex-post IR buyers in the market, the following signaling π^* and fixed price p^* forms an optimal mechanism:*

$$\pi^*(q, s_i) = \begin{cases} 1 & \text{if } i \in \arg\max_i v_i(q) \\ 0 & \text{otherwise} \end{cases}.$$

$$p^* = \mathop{\mathbf{E}}_q [v_{max}(q)].$$

Proof. Similar to the proof of Theorem 4.5, we ignore constraint (17) and focus on the remaining problem. Then we show that the optimal solution also satisfies constraint constraint (17), thus remaining optimal.

First, we construct a signaling scheme as follows:

$$\pi(q, s_i) = \begin{cases} 1 & \text{if } i \in \arg\max_i v_i(q) \\ 0 & \text{otherwise} \end{cases}.$$

It means that for any q, the seller will send signal 1 to the buyer i with the highest $v_i(q)$. Then the obedience constraint (16) becomes:

$$\int_q v_{max}(q) g(q) \,\mathrm{d}q \ge \int_q g(q) \,\mathrm{d}q \cdot p = p$$

So we can set the fixed price as follows:

$$p = \int_q v_{max}(q) g(q) \,\mathrm{d}q$$
$$= \mathop{\mathbf{E}}_q [v_{max}(q)].$$

Next, we show the constructed mechanism (π^*, p^*) also satisfies the omitted constraint (17). Plugging (π^*, p^*) into constraint (17), the right hand side of this constraint becomes:

$$\mathop{\mathbf{E}}_q [v_i(q)] - \mathop{\mathbf{E}}_q [v_{max}(q)] \le 0.$$

Therefore, this constraint is already implied by constraint (16).

This concludes the proof. □

5 SUMMARY

So far we have studied the revenue-maximizing mechanism design problem for an item seller under two different buyers' behavior patterns and two design spaces. Next, we make a summary of all the results, as shown in Table 1.

Note that when buyers are ex-post IR buyers and the mechanism is a fixed-price signaling mechanism, the result has two terms. The first term – means that there is no fixed-price signaling mechanism that is obedient for all buyers. The second term denotes the revenue obtained under the assumption that the buyer cannot buy the item when the seller does not recommend it.

From Table 1 we can draw the following conclusions:

- When there is one buyer in the market, no matter which patterns the buyer adopts, the revenues obtained from two mechanism spaces are the same.
- When there are multiple buyers in the market, and all buyers are ex-interim IR, the revenue generated by the fixed-price signaling mechanism is no less than the revenue generated by the fixed-price mechanism.

The first conclusion implies that the ability to design information does not bring the seller more revenue, which is counter-intuitive. On the contrary, the second conclusion indicates that when multiple ex-interim IR buyers are in the market, the seller will obtain more revenue by designing information before deciding on the fixed price.

6 CONCLUSION

In this paper, we studied the optimal fixed-price signaling mechanism design problem for an item seller in a market with multiple buyers. Throughout the analysis, we considered two types of behavior patterns of the buyer. As a benchmark, we first investigated the optimal fixed-price mechanism without signaling. Moreover, we dived into our main questions and found that when there is only one buyer in the market, the revenues are the same from using both two mechanisms. So in this case, the ability to design information will not bring the seller more benefits. However, when multiple buyers are in the market and all of them are ex-interim IR, the seller will obtain more revenue from using the fixed-price signaling mechanism.

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